Can we constrain the maximum value for the spin parameter of the super-massive objects in galactic nuclei without knowing their actual nature?

Cosimo Bambi*
Arnold Sommerfeld Center for Theoretical Physics, Ludwig-Maximilians-Universität München, 80333 Munich, Germany and Institute for the Physics and Mathematics of the Universe, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan
(Dated: October 11, 2011)

In 4-dimensional General Relativity, black holes are described by the Kerr solution and are subject to the bound $|a_*| \le 1$, where a_* is the black hole spin parameter. If current black hole candidates are not the black holes predicted in General Relativity, this bound does not hold and a_* might exceed 1. In this letter, I relax the Kerr black hole hypothesis and I find that the value of the spin parameter of the super-massive black hole candidates in galactic nuclei cannot be higher than about 1.2. A higher spin parameter would not be consistent with a radiative efficiency $\eta > 0.15$, as observed at least for the most luminous AGN. While a rigorous proof is lacking, I conjecture that the bound $|a_*| \lesssim 1.2$ is independent of the exact nature of these objects.

Introduction — Nowadays there is robust observational evidence of the existence of $5-20\ M_\odot$ dark bodies in X-ray binary systems and of $10^5 - 10^9\ M_\odot$ dark bodies in galactic nuclei [1]. While the estimate of the masses of these objects is reliable, as based on dynamical measurements, we do not know very much about their true nature. The conjecture is that they are the black holes (BHs) predicted in General Relativity. The stellar-mass objects in X-ray binary systems are too heavy to be neutron or quark stars for any reasonable matter equation of state [2]. At least some of the super-massive objects in galactic nuclei are too massive, compact, and old to be clusters of non-luminous bodies, as the cluster lifetime due to evaporation and physical collision would be shorter than the age of these systems [3]. However, constraints on the geometry of the space-time around these objects are weak [4]. For the time being, we have to fully rely on the validity of General Relativity, which is tested only in the weak field limit (Solar System and binary pulsars), where $g_{tt} \approx -(1+2\phi)$ and $|\phi| \lesssim 10^{-6}$ [5].

In 4-dimensional General Relativity, BHs are described by the Kerr solution and are completely specified by two parameters: the mass M and the spin angular momentum J [6]. A fundamental limit for a BH in General Relativity is the Kerr bound $|a_*| \le 1$, where $a_* = J/M^2$ is the spin parameter. This is just the condition for the existence of the event horizon: for $|a_*| > 1$ the event horizon disappears and the central singularity becomes naked, violating the weak cosmic censorship conjecture [7]. Despite the possibility of forming naked singularities from regular initial data [8], the existence of a Kerr naked singularity can be excluded at least for the following reasons: it is apparently impossible to make a star collapse with $|a_*| > 1$ [9] or overspin an already existing BH up to $|a_*| > 1$ [10] and, even if created, a Kerr naked singularity would be highly unstable [11].

On the other hand, if the current BH candidates are not the BHs predicted in General Relativity, the Kerr bound does not hold and the maximum value for a_* may be either larger or smaller than 1, depending on the actual nature of these objects [12, 13]. Generally speaking, bodies with spin parameter larger than 1 are not necessarily monsters: for non-compact objects, a_* can easily exceed 1. For example, the Earth has $a_* \sim 10^3$. In the case of compact objects, a high a_* is more difficult and, for instance, the maximum value of the spin parameter of a neutron star is thought to be about 0.6, because otherwise the object becomes unstable and spins down by emitting gravitational radiation [14]. As shown in [15], if the geometry around a compact object deviates from the Kerr metric, the accretion process can naturally spin the object up to $|a_*| > 1$.

On the basis of these considerations, it is interesting to figure out if current observations can provide some constraint on the maximum value of the spin of the BH candidates, even if we do not know their nature.

Non-Kerr compact objects — At first approximation, a non-Kerr compact object can be described by three parameters: in addition to the mass and the spin angular momentum, we can introduce a "deformation parameter", say ϵ, which measures the deviations from the Kerr geometry. It is convenient that for $\epsilon = 0$ we recover exactly the Kerr solution. In the literature there are a few proposals that can do the job [16]. Here I use the metric recently suggested in [17], because it has the advantage that a_* and ϵ can assume any value. In Boyer-Lindquist coordinates, the metric reads

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2Mr}{\Sigma}\right)(1+h) \, , \\ g_{t\phi} &= -\frac{2aMr\sin^2\theta}{\Sigma}(1+h) \, , \\ g_{rr} &= \frac{\Sigma(1+h)}{\Delta + a^2 h \sin^2\theta} \, , \\ g_{\theta\theta} &= \Sigma \, , \end{aligned}$$

$$g_{\phi\phi} = \left(r^2 + a^2 + \frac{2a^2 M r \sin^2\theta}{\Sigma}\right)\sin^2\theta + \\ + \frac{a^2(\Sigma + 2Mr)\sin^4\theta}{\Sigma} h \,, \tag{1}$$

where $a = a_* M$ and

$$\Sigma = r^2 + a^2\cos^2\theta \,, \\ \Delta = r^2 - 2Mr + a^2 \\ h = \frac{\epsilon M^3 r}{\Sigma^2} \,. \tag{2}$$

The compact object is more prolate (oblate) than a Kerr BH for $\epsilon > 0$ ($\epsilon < 0$); when $\epsilon = 0$, we recover the Kerr solution.

Radiative efficiency — The luminosity of a compact object due to the accretion process is simply $L_{acc} = \eta \dot{M}$, where η is the radiative efficiency and $\dot{M}$ is the mass accretion rate. The value of η depends on the details of the accretion process. For instance, in the case of Bondi accretion onto a Schwarzschild BH, the gas cannot radiate efficiently the energy gained by falling into the BH gravitational potential and $\eta \sim 10^{-4}$ [18]. The highest value of the radiative efficiency is reached when a BH is surrounded by a geometrically thin and optically thick accretion disk. The gas's particles inspiral into the central object by losing energy and angular momentum. When they reach the inner edge of the disk, which can be supposed to be at the innermost stable circular orbit (ISCO), they plunge into the BH. If the gas does not emit additional radiation during the plunging and no radiation is emitted from the surface of the compact object (as it is observed in the case of BH candidates [19]), the radiative efficiency is

$$\eta = 1 - E_{\rm ISCO} \,, \tag{3}$$

where $E_{\rm ISCO}$ is the specific energy of a particle at the ISCO. In the Kerr background, $\eta = 0.057$ for a non-rotating BH, $\eta = 0.32$ for a BH rotating at the Thorne's limit (i.e. $a_* = 0.998$) [20], and $\eta = 0.42$ for an extreme BH (i.e. $a_* = 1$).

For a generic axially symmetric and stationary background, Eq. (3) can be computed as follows. One assumes that the disk is on the equatorial plane and that the gas moves on nearly geodesic circular orbits. In cylindrical coordinates, the equations of the geodesic motion of a particle around the compact object are

$$\dot{t} = \frac{E g_{\phi\phi} + L g_{t\phi}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} \,, \tag{4}$$

$$\dot{\phi} = \frac{E g_{t\phi} + L g_{tt}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} \,, \tag{5}$$

$$g_{rr}\dot{r}^2 + g_{zz}\dot{z}^2 = V_{\rm eff}(E, L, r, z) \,, \tag{6}$$

where E and L are respectively the conserved specific energy and the conserved specific z-component of the angular momentum, while $V_{\rm eff}$ is the effective potential

$$V_{\rm eff} = \frac{E^2 g_{\phi\phi} + 2EL g_{t\phi} + L^2 g_{tt}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} - 1 \,. \tag{7}$$

Circular orbits on the equatorial plane are located at the zeros and the turning points of the effective potential: $\dot{r} = \dot{z} = 0$ implies $V_{\rm eff} = 0$, and $\ddot{r} = \ddot{z} = 0$ requires $\partial_r V_{\rm eff} = \partial_z V_{\rm eff} = 0$. From these conditions, we can get E and L:

$$E = -\frac{g_{tt} + g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}} \,, \tag{8}$$

$$L = \frac{g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}} \,, \tag{9}$$

where

$$\Omega = \frac{-\partial_r g_{t\phi} \pm \sqrt{(\partial_r g_{t\phi})^2 - (\partial_r g_{tt})(\partial_r g_{\phi\phi})}}{\partial_r g_{\phi\phi}} \tag{10}$$

is the orbital angular velocity and the sign $+$ ($-$) is for corotating (counterrotating) orbits. The orbits are stable under small perturbation if $\partial_r^2 V_{\rm eff} \le 0$ and $\partial_z^2 V_{\rm eff} \le 0$. One can thus find numerically the ISCO radius and get the specific energy $E_{\rm ISCO}$ and the maximum efficiency parameter $\eta = 1 - E_{\rm ISCO}$ for any value of a_* and ϵ.

Fig. 1 shows some contours of the radiative efficiency for an object with spin parameter a_* and deformation parameter ϵ for the metric (1). The radiative efficiency is $\eta = 0.15$ (red solid curve), $\eta = 0.20$ (green dashed curve), and $\eta = 0.25$ (blue dotted curve).

Evolution of the spin parameter — The value of the spin parameter of a compact object is determined by the competition of three physical processes: the event creating the object, mergers, and gas accretion. For the super-massive objects in galactic nuclei, independently of their nature, the initial value of the spin parameter is completely irrelevant: their mass has increased by several orders of magnitude from its original value, and the spin parameter has evolved accordingly. On average, the capture of small bodies (minor merger) in randomly oriented orbits spins any compact object down, since the magnitude of the orbital angular momentum for corotating orbits is always smaller than the one for counterrotating orbits [21]. The case of coalescence of two compact objects with comparable mass (major merger) can be rigorously computed only if we know the exact nature of these objects and the theory of gravity, as the background is not fixed and the emission of gravitational waves may be important. In General Relativity, the product of the merger of two neutron stars is a black hole with $a_* \approx 0.78$, depending only weakly on the total mass and mass ratio of the system [22]. In the case of random merger of two black holes, the most probable final product is a

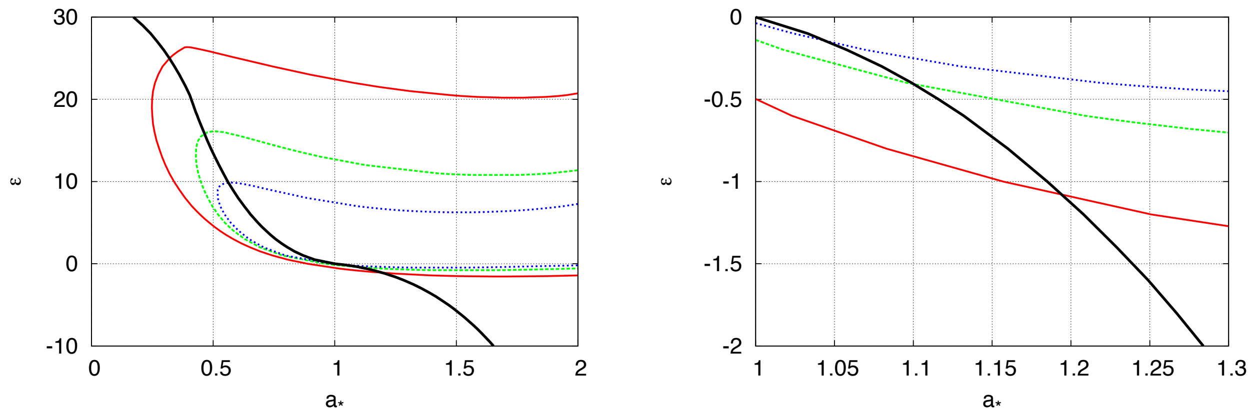

FIG. 1. Compact objects with spin parameter a_* and deformation parameter ϵ. The radiative efficiency is $\eta = 0.15$ (red solid curve), $\eta = 0.20$ (green dashed curve), and $\eta = 0.25$ (blue dotted curve). The black solid curve is the equilibrium spin parameter a_*^{eq} obtained from Eq. (11). The right panel is an enlargement of the parameter region $1.0 < a_* < 1.3$ and $-2.0 < \epsilon < 0.0$.

black hole with $a_* \approx 0.70$, while fast-rotating object with $a_* > 0.9$ should be rare [23].

Accretion from a disk can potentially be a very efficient way to spin a compact object up [23]. If the inner edge of the disk is at the ISCO radius, the gas's particles plunge into the compact object with specific energy $E_{\rm ISCO}$ and specific angular momentum $L_{\rm ISCO}$. The mass M and the spin angular momentum J of the compact object change respectively by $\delta M = E_{\rm ISCO} \delta m$ and $\delta J = L_{\rm ISCO} \delta m$, where δm is the gas rest-mass. The evolution of the spin parameter is governed by the following equation [24]

$$\frac{da_*}{d \ln M} = \frac{1}{M} \frac{L_{\rm ISCO}}{E_{\rm ISCO}} - 2a_* \, , \tag{11}$$

neglecting the small effect of the radiation emitted by the disk and captured by the object. If accretion proceeds via short episodes (chaotic accretion) [25], the net effect is not different from minor mergers in randomly oriented orbits and the compact object is spun down. On the contrary, prolonged disk accretion is a very efficient mechanism to spin the compact object up, till an equilibrium spin parameter a_*^{eq}, which is reached when the right-hand side of Eq. (11) becomes zero. For instance, an initially non-rotating Kerr BH reaches the equilibrium $a_*^{eq} = 1$ after having increased its mass by a factor $\sqrt{6} \approx 2.4$ [24].

We can thus say that the most optimistic scenario to produce fast-rotating super-massive objects at the center of galaxies is via prolonged disk accretion and that the maximum value for the spin parameter of these objects cannot exceed a_*^{eq}. The numerical value of a_*^{eq} depends on the metric of the space-time. The black solid curve in Fig. 1 shows the equilibrium spin parameter for the metric (1). Objects on the left of the black solid curve have $a_* < a_*^{eq}$ and the accretion process spins them up; objects on the right have $a_* > a_*^{eq}$ and the accretion process spins them down. As already noted in Ref. [15], objects more oblate than a BH (for the metric (1) when $\epsilon < 0$) have $a_*^{eq} > 1$.

Observational constraints — In general, it is not easy to get an estimate of η, as the measurement of the mass accretion rate $\dot{M}$ is typically quite problematic. The mean radiative efficiency of AGN can be inferred from the Soltan's argument [26], which relates the mass density of the super-massive BH candidates in the contemporary Universe with the energy density of the radiation produced in the whole history of the Universe by the accretion process onto these objects. There are several sources of uncertainty in the final result, but a mean radiative efficiency $\eta > 0.15$ seems to be a conservative lower limit [27]. The authors of Ref. [28] find a mean radiative efficiency $\eta \approx 0.30 - 0.35$ without some important assumptions necessary in the original version of the Soltan's argument. In Ref. [29], the authors show how to estimate η for individual AGN and find that the more massive objects have typically higher η, up to $\sim 0.3 - 0.4$.

Here, it is not important the mean radiative efficiency of these objects. It is sufficient to say that at least some of them must have $\eta > 0.15$. In other words, the space-time around the super-massive BH candidates allows for a specific energy at the ISCO radius smaller than 0.85. This fact is non-trivial, as it says that the ISCO radius can be quite close to the object (= gravity cannot be too strong). On the other hand, very high spin parameters could be possible only in stronger gravitational fields, in which the ISCO radius is larger and $L_{\rm ISCO}/E_{\rm ISCO}$ is larger too. So, if we use the metric (1) to describe the geometry of the space-time around the super-massive BH candidates in galactic nuclei, we find that the spin parameter of these objects cannot exceed 1.19, see Fig. 1. The maximum value for $|a_*|$ becomes 1.10 if we require $\eta > 0.20$, and 1.04 for $\eta > 0.25$.

Comments — To show that the result $|a_*| \lesssim 1.2$ seems to be robust, it is necessary to address at least two points,

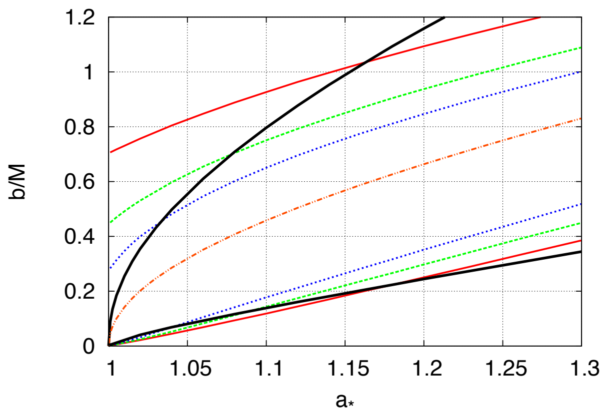

FIG. 2. Compact objects with spin parameter a_* and deformation parameter b described by the Manko-Mielke-Sanabria Gomez (MMS) solution. The radiative efficiency is $\eta = 0.15$ (red solid curve), $\eta = 0.20$ (green dashed curve), and $\eta = 0.25$ (blue dotted curve). The black solid curve is the equilibrium spin parameter a_*^{eq} obtained from Eq. (11). For $b = M\sqrt{a_*^2 - 1}$ (orange dashed-dotted curve), we recover the Kerr solution, which is in the region $a_* > a_*^{eq}$.

concerning respectively its dependence on the choice of the metric (1) and the validity of Eqs. (3) and (11).

The bound $|a_*| \lesssim 1.2$ seems to depend only marginally on the choice of the metric. For instance, we get quite similar constraints if we consider the Manko-Mielke-Sanabria Gomez (MMS) metric, which is an exact solution of the Einstein's vacuum equation (the metric (1) is not) and does not describe a BH (while the metric (1) does); see the second paper in [15]. In addition to the mass and the spin parameter, the MMS solution has a deformation parameter b. When $|a_*| \geq 1$, the Kerr metric is recovered for $b = \pm M\sqrt{a_*^2 - 1}$; around $b = M\sqrt{a_*^2 - 1}$ there are objects more oblate than Kerr BHs, around $b = -M\sqrt{a_*^2 - 1}$ the objects are more prolate than Kerr BHs. The constraints on the maximum value for the spin parameter are shown in Fig. 2 – here I show only the parameter space $b > 0$ because for more prolate objects we find lower values. The bounds turn out to be 1.18 if we require $\eta > 0.15$, 1.09 for $\eta > 0.20$, and 1.04 for $\eta > 0.25$. Despite the different nature of the two metrics, it is remarkable that we get very similar constraints. The point is that the constraint on the maximum value for the spin parameter is not very sensitive to the exact space-time geometry, but it depends on how much the compact object is more or less oblate.

The result relies also on the validity of Eqs. (3) and (11). As discussed in Ref. [30], in backgrounds deviating from the Kerr geometry, the gas may not plunge from the ISCO into the central object; if this is the case, the gas must form a thick disk inside the ISCO radius and lose additional energy and angular momentum. That increases the radiative efficiency at most by a few percent with respect to the value calculated from Eq. (3). It also slightly decreases the equilibrium parameter a_*^{eq}, as the gas plunges from a radius inside the ISCO. However, for the metric (1) and $\epsilon < 0$ such a possibility never happens: accretion proceeds as in the Kerr space-time and the result $|a_*| \lesssim 1.2$ is not affected.

Lastly, let us consider the possibility that the initial value of the spin parameter of the object is $a_*^{in} > a_*^{eq}$. In this case, the accretion process would spin the object down, approaching a_*^{eq} from the right of the black solid line in Fig. 1, but the bound $|a_*| \lesssim 1.2$ can still be applied. The initial value of the spin parameter of the super-massive objects in galactic nuclei is presumably negligible: their mass has increased by several orders of magnitude from its original value and a_* has evolved according to Eq. (11). If a_*^{in} were of order 1, a_*^{eq} was reached soon, after the object increased its mass by a factor of order 1. The possibility that this gravity theory can make a star collapse with $|a_*| \gg 1$ and that the super-massive black hole candidates have still a spin parameter significantly larger than a_*^{eq} seems to be very unlikely, at least for two reasons. The accretion process onto an object with $|a_*| \gg 1$ is strongly suppressed and the object could have not become super-massive [13]. This behavior does not depend on the exact metric of the space-time, because the effect of the spin would be important relatively far from the object, where deviations from the Kerr geometry are more suppressed. The second reason is that compact objects with spin parameter $|a_*| \gg 1$ are usually unstable. For instance, Ref. [11] shows that the Kerr metric with $|a_*| > 1$ is unstable because of the existence of stable photon orbits with negative energy and that this conclusion does not depend on the exact gravity theory.

Conclusions — In 4-dimensional General Relativity, BHs are subject to the bound $|a_*| \leq 1$, where $a_* = J/M^2$ is the spin parameter. If the current BH candidates are not the BH predicted in General Relativity, this bound does not hold and a_* might exceed 1. In this letter, I have relaxed the common assumption that the super-massive objects at the center of every normal galaxy are Kerr BHs and I have found that current observations can constrain the maximum value of the spin parameter of these object at the level of $|a_*| \lesssim 1.2$. While I cannot provided a rigorous proof, my conjecture is that this bound holds whatever the nature of these objects is. The origin of this bound can be heuristically understood as follows. A higher spin parameter would require a larger ISCO radius: both $L_{\rm ISCO}$ and $E_{\rm ISCO}$ decrease as the ISCO radius decreases, but $L_{\rm ISCO}$ decreases faster. However, a larger ISCO radius implies a lower maximum radiative efficiency $\eta_{max} = 1 - E_{\rm ISCO}$, while we know that at least some of the super-massive objects in galactic nuclei must have $\eta > 0.15$. From the latter, we get $|a_*| \lesssim 1.2$.

This work was supported by World Premier International Research Center Initiative (WPI Initiative),

MEXT, Japan, by the JSPS Grant-in-Aid for Young Scientists (B) No. 22740147, and by Humboldt Foundation.

---

* Cosimo.Bambi@physik.uni-muenchen.de

[1] R. Narayan, New J. Phys. **7**, 199 (2005). [gr-qc/0506078].

[2] V. Kalogera and G. Baym, Astrophys. J. **470**, L61 (1996).

[3] E. Maoz, Astrophys. J. **494**, L181 (1998).

[4] C. Bambi, arXiv:1109.4256 [gr-qc]; C. Bambi, E. Barausse, Astrophys. J. **731**, 121 (2011). [arXiv:1012.2007 [gr-qc]]; C. Bambi, Phys. Rev. **D83**, 103003 (2011). [arXiv:1102.0616 [gr-qc]].

[5] C. M. Will, Living Rev. Rel. **9**, 3 (2005). [gr-qc/0510072].

[6] B. Carter, Phys. Rev. Lett. **26**, 331-333 (1971); D. C. Robinson, Phys. Rev. Lett. **34**, 905-906 (1975); P. T. Chrusciel, J. Lopes Costa, arXiv:0806.0016 [gr-qc].

[7] R. Penrose, Riv. Nuovo Cim. **1**, 252-276 (1969).

[8] P. S. Joshi, D. Malafarina, Phys. Rev. **D83**, 024009 (2011). [arXiv:1101.2084 [gr-qc]]; P. S. Joshi, D. Malafarina, arXiv:1105.4336 [gr-qc].

[9] B. Giacomazzo, L. Rezzolla, N. Stergioulas, Phys. Rev. **D84**, 024022 (2011). [arXiv:1105.0122 [gr-qc]].

[10] E. Barausse, V. Cardoso, G. Khanna, Phys. Rev. Lett. **105**, 261102 (2010). [arXiv:1008.5159 [gr-qc]].

[11] P. Pani, E. Barausse, E. Berti, V. Cardoso, Phys. Rev. **D82**, 044009 (2010). [arXiv:1006.1863 [gr-qc]].

[12] C. Bambi, K. Freese, Phys. Rev. **D79**, 043002 (2009). [arXiv:0812.1328 [astro-ph]].

[13] C. Bambi, K. Freese, T. Harada, R. Takahashi, N. Yoshida, Phys. Rev. **D80**, 104023 (2009). [arXiv:0910.1634 [gr-qc]]; C. Bambi, T. Harada, R. Takahashi, N. Yoshida, Phys. Rev. **D81**, 104004 (2010). [arXiv:1003.4821 [gr-qc]]; C. Bambi, N. Yoshida, Phys. Rev. **D82**, 064002 (2010). [arXiv:1006.4296 [gr-qc]]; C. Bambi, N. Yoshida, Phys. Rev. **D82**, 124037 (2010). [arXiv:1009.5080 [gr-qc]].

[14] N. Andersson, D. I. Jones, K. D. Kokkotas and N. Stergioulas, Astrophys. J. **534**, L75 (2000).

[15] C. Bambi, Europhys. Lett. **94**, 50002 (2011). [arXiv:1101.1364 [gr-qc]]; C. Bambi, JCAP **1105**, 009 (2011). [arXiv:1103.5135 [gr-qc]]; C. Bambi, [arXiv:1104.2218 [gr-qc]].

[16] V. S. Manko and I. D. Novikov, Class. Quant. Grav. **9**, 2477 (1992); K. Glampedakis, S. Babak, Class. Quant. Grav. **23**, 4167-4188 (2006). [gr-qc/0510057]. S. J. Vigeland, S. A. Hughes, Phys. Rev. **D81**, 024030 (2010). [arXiv:0911.1756 [gr-qc]].

[17] T. Johannsen, D. Psaltis, arXiv:1105.3191 [gr-qc].

[18] S. L. Shapiro and S. A. Teukolsky, *Black holes, white dwarfs, and neutron stars: The physics of compact objects*, (Wiley, New York, New York, 1983).

[19] J. E. McClintock, R. Narayan, G. B. Rybicki, Astrophys. J. **615**, 402-415 (2004). [astro-ph/0403251]; A. E. Broderick, A. Loeb, R. Narayan, Astrophys. J. **701**, 1357-1366 (2009). [arXiv:0903.1105 [astro-ph.HE]].

[20] K. S. Thorne, Astrophys. J. **191**, 507 (1974).

[21] S. A. Hughes, R. D. Blandford, Astrophys. J. **585**, L101-L104 (2003). [astro-ph/0208484]; C. F. Gammie, S. L. Shapiro, J. C. McKinney, Astrophys. J. **602**, 312-319 (2004). [astro-ph/0310886].

[22] K. Kiuchi, Y. Sekiguchi, M. Shibata, K. Taniguchi, Phys. Rev. **D80**, 064037 (2009). [arXiv:0904.4551 [gr-qc]].

[23] E. Berti and M. Volonteri, Astrophys. J. **684**, 822 (2008).

[24] J. M. Bardeen, Nature **226**, 64 (1970).

[25] A. R. King, J. E. Pringle, Mon. Not. Roy. Astron. Soc. Lett. **373**, L93-L97 (2006). [astro-ph/0609598].

[26] A. Soltan, Mon. Not. Roy. Astron. Soc. **200**, 115-122 (1982).

[27] M. Elvis, G. Risaliti, G. Zamorani, Astrophys. J. **565**, L75-L77 (2002). [astro-ph/0112413].

[28] J. -M. Wang, Y. -M. Chen, L. C. Ho, R. J. McLure, Astrophys. J. **642**, L111-L114 (2006). [astro-ph/0603813].

[29] S. W. Davis, A. Laor, Astrophys. J. **728**, 98 (2011). [arXiv:1012.3213 [astro-ph.CO]].

[30] C. Bambi, E. Barausse, arXiv:1108.4740 [gr-qc].